

## THE EQUIVARIANT COVERING HOMOTOPY PROPERTY FOR DIFFERENTIABLE $G$ -FIBRE BUNDLES

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Let  $G$  be a compact Lie group, and  $X$  a differentiable  $G$ -manifold. If  $p: E \rightarrow X$  is a differentiable fibre bundle, and  $G$  acts differentiably on  $E$  so that each  $g \in G$  operates as a bundle map, then we call  $p$  a differentiable  $G$ -fibre bundle. We show that if  $p$  is a differentiable  $G$ -fibre bundle with Lie structure group or compact fibre, then it has the equivariant covering homotopy property. This generalizes the fact that a differentiable family of actions of a compact Lie group on a compact differentiable manifold is locally trivial.

We give some basic definitions in § 1, and in § 2 show that if  $X$  is a  $G$ -manifold and  $E \rightarrow X$  a differentiable fibre bundle with Lie structure group  $H$  and associated principal bundle  $P \rightarrow X$ , then differentiable actions of  $G$  on  $E$  as a group of bundle maps are in natural one-one correspondence with such actions on  $P$ . In § 3 we establish the equivariant covering homotopy property for differentiable  $G$ -fibre bundles with compact Lie structure group, and show that if  $p: E \rightarrow X$  is a differentiable  $G$ -fibre bundle with connected semi-simple Lie structure group  $H$ , then  $p$  can be reduced to a compact subgroup of  $H$  so that  $G$  still operates as a group of bundle maps, and hence  $p$  also has the equivariant covering homotopy property. Then in § 4 we define a notion of equivariant local triviality for  $G$ -fibre bundles, which implies the equivariant covering homotopy property, and show that any differentiable  $G$ -fibre bundle with Lie structure group or compact fibre is  $G$ -locally trivial. We conclude with some remarks relating  $G$ -local triviality to the equivalence of nearby differentiable actions of a compact Lie group.

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### 1. Basic definitions

Let  $G$  be a topological group. A  $G$ -space is a Hausdorff space  $X$  together with a *continuous action* of  $G$  on  $X$ , i.e., a continuous map  $(g, x) \rightarrow gx$  of  $G \times X$  into  $X$  such that  $g_1(g_2x) = (g_1g_2)x$  for all  $g_1, g_2 \in G$ ,  $x \in X$ , and  $1x = x$ ,

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where 1 is the identity element of  $G$ . If  $G$  is a Lie group, then a (*differentiable*)  $G$ -manifold is a differentiable ( $C^\infty$ ) manifold  $X$  together with a *differentiable action* of  $G$  on  $X$ . The action is *effective* if whenever  $gx = x$  for some  $g$  and all  $x$ , then  $g = 1$ .

Let  $G$  be a compact Lie group, and  $X$  a differentiable  $G$ -manifold. Let  $G_x$  be the isotropy subgroup of a point  $x \in X$ . The map  $G/G_x \rightarrow X$  defined by  $gG_x \rightarrow gx$  is an equivariant embedding whose image is the orbit  $Gx$ . Let  $V_x = TX_x/T(Gx)_x$  be the normal space to the orbit  $Gx$  at the point  $x$ . For  $g \in G_x$ , the differential of  $g: X \rightarrow X$  induces an automorphism of  $V_x$ , so we have a representation  $G_x \rightarrow GL(V_x)$ , called a *slice representation*. The *slice bundle*  $G \times_{G_x} V_x$  is the  $G$ -vector bundle constructed from the product  $G \times V_x$  by identifying  $(gh, h^{-1}v)$  with  $(g, v)$  for all  $g \in G$ ,  $h \in G_x$ ,  $v \in V_x$ ; we let  $[g, v]$  denote the image of  $(g, v)$  in  $G \times_{G_x} V_x$  under the identification map. Using the identification  $G/G_x \rightarrow Gx$ , we can identify the slice bundle  $G \times_{G_x} V_x$  with the normal bundle of  $Gx$  in  $X$  by the map  $[g, v] \rightarrow gv$ . Hence, by an equivariant version of the tubular neighborhood theorem, there is an equivariant diffeomorphism from  $G \times_{G_x} V_x$  onto a  $G$ -invariant open neighborhood of  $Gx$  in  $X$ , mapping the zero section  $G/G_x$  canonically onto the orbit  $Gx$ . We call the image of  $V_x$  a *slice* at  $x$ .

A *fibre bundle* is a continuous map  $p: E \rightarrow X$  of a Hausdorff space  $E$  onto a Hausdorff space  $X$  such that  $p$  is locally trivial. Let  $p_i: E_i \rightarrow X_i$ ,  $i = 1, 2$ , be two fibre bundles. A *bundle map* from  $p_1$  to  $p_2$  is a continuous map  $F: E_1 \rightarrow E_2$  which carries each fibre homeomorphically onto a fibre. The induced map  $f: X_1 \rightarrow X_2$  is clearly continuous. If  $X_1 = X_2$  and the induced map is the identity (so that  $F$  is a homeomorphism), then the bundle map is called an *equivalence*.

We also consider fibre bundles with a specified structure group which acts effectively on the typical fibre. When a structure group is specified, bundle maps are understood to be induced by principal bundle maps between the associated principal bundles. Note that if  $p_i: E_i \rightarrow X_i$ ,  $i = 1, 2$ , are fibre bundles with structure group the identity and fibre  $Y$ , then  $E_i$  is equivalent to  $X_i \times Y$ ,  $i = 1, 2$ , and over each map  $f: X_1 \rightarrow X_2$  of the bases there is only one bundle map, corresponding to  $f \times \text{id}: X_1 \times Y \rightarrow X_2 \times Y$ .

We will mainly be concerned with differentiable ( $C^\infty$ ) fibre bundles. In this case the spaces are differentiable manifolds, the maps are  $C^\infty$ , and a structure group is a Lie group acting differentiably (from the left) on the typical fibre (and so differentiably from the right on the total space of the associated principal bundle).

## 2. $G$ -fibre bundles

In the remainder of this paper  $G$  will denote a compact Lie group.

Let  $X$  be a  $G$ -space, and  $p: E \rightarrow X$  a fibre bundle over  $X$ . If there is a

continuous action of  $G$  on  $E$  such that each  $g \in G$  operates as a bundle map over the given map  $g: X \rightarrow X$  (hence, in the case that  $p$  has a specified structure group, is induced by a principal bundle map), then we say that  $G$  acts on  $p: E \rightarrow X$  as a group of bundle maps and that  $p$  is a  $G$ -fibre bundle (differentiable if  $p$  is a differentiable fibre bundle and  $X, E$  are  $G$ -manifolds). Note that  $p$  is equivariant. A  $G$ -fibre bundle map (resp.  $G$ -fibre bundle equivalence) is a map (resp. equivalence) of  $G$ -fibre bundles which is equivariant with respect to the actions of  $G$ .

**Example 1.** Let  $X, Y$  be  $G$ -spaces.  $G$  acts on  $X \times Y$  by  $g(x, y) = (gx, gy)$  for  $g \in G, (x, y) \in X \times Y$ . The projection  $p: X \times Y \rightarrow X$  is equivariant, and  $G$  acts as a group of bundle maps if we consider  $p$  as a trivial fibre bundle with structure group  $G$ . We call  $p$  a trivial  $G$ -fibre bundle.

**Example 2.** A  $G$ -vector bundle is a  $G$ -fibre bundle with structure group a general linear group. The results in this paper are given by Segal [7] for  $G$ -vector bundles over compact spaces, and by Wasserman [8] for differentiable  $G$ -vector bundles over  $G$ -manifolds.

**Proposition 2.1.** Let  $p_i: E_i \rightarrow X_i, i = 1, 2$ , be  $G$ -fibre bundles with the same structure group and fibre. If  $f: X_1 \rightarrow X_2$  is equivariant, then the induced bundle  $f^*E_2$  over  $X_1$  is naturally a  $G$ -fibre bundle, and the induced map  $f^*E_2 \rightarrow E_2$  is a  $G$ -fibre bundle map. If  $F: E_1 \rightarrow E_2$  is a  $G$ -fibre bundle map over  $f$ , then  $E_1$  is  $G$ -equivalent to  $f^*E_2$ , and  $F$  is the composition of a  $G$ -equivalence  $E_1 \rightarrow f^*E_2$  and the induced map  $f^*E_2 \rightarrow E_2$ .

The proof is clear.

Now let  $P \rightarrow X$  be a differentiable principal bundle with structure group a Lie group  $H$ . Let  $Y$  be an effective  $H$ -manifold, and  $E = P \times_H Y \rightarrow X$  the bundle with fibre  $Y$  associated to  $P$ . In other words  $E$  is obtained from the product  $P \times Y$  by identifying  $(p, y)$  with  $(ph, h^{-1}y)$  for all  $p \in P, y \in Y, h \in H$ , and the projection  $E \rightarrow X$  is induced by the projection  $P \rightarrow X$ . Since  $H$  acts effectively on  $Y$ , there is a one-one correspondence between actions of  $G$  as a group of bundle maps of  $E \rightarrow X$  and actions as a group of bundle maps of  $P \rightarrow X$ ; we just take, for the operation of each element of  $G$  as a bundle map  $E \rightarrow X$ , the associated bundle map from  $P \rightarrow X$  to itself, and vice-versa. If  $G$  acts differentiably as a group of bundle maps of  $P \rightarrow X$ , then the induced action on  $E \rightarrow X$  is differentiable. Conversely, we have

**Theorem 2.2.** If  $G$  acts differentiably as a group of bundle maps of  $E \rightarrow X$ , then the induced action on the associated principal bundle  $P \rightarrow X$  is differentiable.

*Proof.* Let  $g_0 \in G, x_0 \in X$ . Choose neighborhoods  $U$  of  $x_0$  in  $X, V$  of  $g_0x_0$  in  $X$ , and  $W$  of  $g_0$  in  $G$  such that the bundle  $E \rightarrow X$  is trivial over  $U$  and  $V$ , and  $W \cdot U \subseteq V$ . With respect to trivialisations  $U \times Y, V \times Y$  of  $E = P \times_H Y$  over  $U, V$ , the action of elements of  $G$  contained in  $W$  is given by a  $C^\infty$  map  $W \times U \times Y \rightarrow V \times Y$ , taking  $(g, u, y) \in W \times U \times Y$  into  $(gu, \alpha(u, g)y)$ , where  $\alpha(u, g) \in H$ . We must show that the map  $\alpha: U \times W \rightarrow H$  is  $C^\infty$ .

Since  $H$  acts effectively on  $Y$ , there is a finite subset  $\{y_1, \dots, y_n\}$  of  $Y$  such that the Lie subgroup  $\{h \in H \mid hy_i = y_i, i = 1, \dots, n\}$  of  $H$  is zero-dimensional (see Gleason and Palais [2, Th. 8. 2, p. 646]). Let  $H$  act on  $Z = Y \times \dots \times Y$  ( $n$  copies) by the given action on each factor, and let  $z = (y_1, \dots, y_n) \in Z$ . Then the isotropy subgroup  $H_z$  is zero-dimensional. Now the map  $U \times W \rightarrow Z$  taking  $(u, g) \in U \times W$  into  $\alpha(u, g)z \in Z$  is  $C^\infty$ , and the image of this map lies in the orbit  $H_z$ , which is diffeomorphic to  $H/H_z$ . In other words, the map  $U \times W \rightarrow H/H_z$  taking  $(u, g)$  to  $\alpha(u, g)H_z$  is  $C^\infty$ . Since  $H_z$  is zero-dimensional, then  $\alpha: U \times W \rightarrow H$  is  $C^\infty$ .

### 3. The equivariant covering homotopy property for differentiable $G$ -fibre bundles reducible to a compact Lie structure group

Using Theorem 2.2 we can prove the following theorem and corollary in the same way they are proved by Wasserman [8, Th. 2.4, Cor. 2.5, p. 134] in the case where the structure group is an orthogonal group.  $I$  denotes the interval  $[0, 1]$ , and  $G$  always acts trivially on  $I$ .

**Theorem 3.1.** *Let  $G$  be a compact Lie group, and  $E \rightarrow X \times I$  a differentiable  $G$ -fibre bundle with structure group a compact Lie group  $H$ . Then there is a differentiable  $G$ -fibre bundle equivalence  $E \rightarrow (E|X \times 0) \times I$ .*

**Corollary 3.2.** *If  $E \rightarrow X$  is a differentiable  $G$ -fibre bundle with compact Lie structure group, and  $f_0, f_1: Y \rightarrow X$  are  $G$ -homotopic (resp. differentiably  $G$ -homotopic)  $G$ -maps from a differentiable  $G$ -manifold  $Y$  to  $X$ , then the induced bundles  $f_0^*E$  and  $f_1^*E$  are  $G$ -equivalent (resp. differentiably  $G$ -equivalent).*

Using Corollary 3.2, we easily deduce the following equivariant covering homotopy property for differentiable  $G$ -fibre bundles with compact Lie structure group.

**Corollary 3.3.** *Let  $E_i \rightarrow X_i, i = 1, 2$ , be differentiable  $G$ -fibre bundles having the same fibre and structure group, a compact Lie group. Let  $F_0: E_1 \rightarrow E_2$  be a  $G$ -fibre bundle map over  $f_0: X_1 \rightarrow X_2$ , and  $f: X_1 \times I \rightarrow X_2$  be a  $G$ -homotopy of  $f_0$ . Then there is a  $G$ -homotopy of  $F_0$ , which is a  $G$ -fibre bundle map  $F: E_1 \times I \rightarrow E_2$  over  $f$ . Moreover, if  $F_0$  is differentiable and  $f$  is a differentiable homotopy, then there is a differentiable covering homotopy  $F$ .*

The above results clearly hold as well for any differentiable  $G$ -fibre bundle whose structure group can be reduced to a compact Lie group so that  $G$  still acts as a group of bundle maps on the reduced bundle. The following theorem then shows that a differentiable  $G$ -fibre bundle whose structure group is a connected semi-simple Lie group has the equivariant covering homotopy property.

**Theorem 3.4.** *Let  $G$  be a compact Lie group, and  $E \rightarrow X$  a differentiable  $G$ -fibre bundle with structure group a connected semi-simple Lie group  $H$ . Then the structure group of  $E \rightarrow X$  can be reduced to a compact subgroup of  $H$  so that  $G$  still acts as a group of bundle maps on the reduced bundle.*

**Remark.** In the case that  $H$  is a general linear group, such a reduction to

the orthogonal group can be given by a  $G$ -invariant Riemannian metric on the associated vector bundle.

*Proof of Theorem 3.4.* Let  $\pi: P \rightarrow X$  be the principal bundle associated with  $E \rightarrow X$ , and  $K$  a maximal compact subgroup of  $H$  (assume  $H$  is not compact). It suffices to find a  $G$ -equivariant section of the bundle  $P/K = P \times_H (H/K) \rightarrow X$ .

The homogeneous space  $H/K$  with any  $H$ -invariant Riemannian metric is a complete simply-connected Riemannian manifold of negative curvature, so that for each  $h \in H$  the exponential map at  $hK \in H/K$  is a diffeomorphism from the tangent space  $T(H/K)_{hK}$  onto  $H/K$  (Helgason [3, Chap. I, Th. 13.3]). Now  $P \times_H T(H/K)$  is a  $G$ -vector bundle over  $P/K$ , called the tangent bundle along the fibres, and the exponential map for  $T(H/K)$  induces a  $G$ -equivariant map  $P \times_H T(H/K) \rightarrow P/K$ , taking  $[ph, h^{-1}v]$  (where  $p \in P$ ,  $h \in H$ ,  $v \in T(H/K)$ ) into  $[ph, \exp(h^{-1}v)] = [ph, h^{-1} \exp v]$ .

For each  $x \in X$  the isotropy subgroup  $G_x$  acts on the fibre  $H/K$  of  $P/K$  over  $x$  via a homomorphism  $G_x \rightarrow H$ . Since all maximal compact subgroups of  $H$  are conjugate (Helgason [3, Chap. VI, Th. 2.2]), the image of this homomorphism is contained in  $hKh^{-1}$  for some  $h \in H$ , so that  $hK$  is a fixed point for the action of  $G_x$  on  $H/K$ . Since  $\pi$  induces a submersion of  $P/K$  onto  $X$ , there is a  $G_x$ -equivariant section  $\sigma$  of  $P/K$  defined on some slice  $V_x$  for  $X$  at  $x$ , with  $\sigma(x) = hK$ . We have then a pull-back diagram:

$$\begin{array}{ccc} \sigma^*(P \times_H T(H/K)) & \longrightarrow & P \times_H T(H/K) \\ \downarrow & & \downarrow \\ V_x & \longrightarrow & P/K \end{array}$$

so that  $\sigma^*(P \times_H T(H/K))$  is a  $G_x$ -vector bundle over  $V_x$ , and the exponential map  $\sigma^*(P \times_H T(H/K)) \rightarrow (P/K)|_{V_x}$  is a  $G_x$ -equivariant fibre-preserving diffeomorphism.

We now construct a  $C^\infty$  equivariant section of  $P/K \rightarrow X$ . For each  $x \in X$ , shrink  $V_x$  equivariantly to  $U_x$ ,  $\text{Cl}(U_x) \subset V_x$  ( $\text{Cl} = \text{closure}$ ), and choose a countable number of points  $x(1), x(2), \dots$  such that the slice neighborhoods  $G \cdot U_{x(i)}$  of the orbits  $Gx(i)$  cover  $X$ . Set  $A_0 = \emptyset$ , and define  $A_n$  inductively by  $A_n = G \cdot \text{Cl}(U_{x(n)}) \cup A_{n-1}$ . Suppose  $C^\infty$  equivariant sections  $s_i$  of  $P/K \rightarrow X$  are defined on  $A_i$  for  $i < n$ , such that  $s_i|_{A_{i-1}} = s_{i-1}$ . Since there is a  $G_{x(n)}$ -equivariant fibre-preserving diffeomorphism from  $(P/K)|_{V_{x(n)}}$  to a  $G_{x(n)}$ -vector bundle over  $V_{x(n)}$ ,  $s_{n-1}$  extends to a  $C^\infty$  equivariant section  $s_n$  over  $A_n$ . Define  $s$  by  $s(x) = s_n(x)$  for  $x \in A_n - A_{n-1}$ . Since  $X$  is the union of the interiors of the  $A_n$ , we see  $s$  is a  $C^\infty$  equivariant section  $X \rightarrow P/K$ .

#### 4. $G$ -local triviality and the equivariant covering homotopy property

Let  $p: E \rightarrow X$  be a differentiable  $G$ -fibre bundle. We say  $p$  is  $G$ -locally trivial

if for each  $x \in X$  there is a  $G_x$ -invariant neighborhood  $U_x$  of  $x$  such that  $p|U_x$  is differentiably  $G_x$ -equivalent to the trivial  $G_x$ -fibre bundle  $U_x \times p^{-1}(x)$ .

By the equivariant covering homotopy property (Corollary 3.3), a differentiable  $G$ -fibre bundle with structure group a compact or semi-simple Lie group is  $G$ -locally trivial. On the other hand, the following theorem implies that if  $p: E \rightarrow X$  is a differentiable  $G$ -fibre bundle which is  $G$ -locally trivial, then  $p$  has the equivariant covering homotopy property.

**Theorem 4.1.** *Let  $G$  be a compact Lie group, and  $p: E \rightarrow X \times I$  a differentiable  $G$ -fibre bundle which is  $G$ -locally trivial ( $G$  acts trivially on  $I$ ). Then there is a differentiable  $G$ -fibre bundle equivalence  $E \rightarrow (E|X \times 0) \times I$  (the map is understood to be induced by a principal bundle map in the case that  $p$  is a  $G$ -fibre bundle with Lie structure group  $H$ ).*

*Proof.* The proof is similar to that of the equivariant covering homotopy property for locally trivial fibre spaces given, for example, in Husemoller [4, pp. 49–51]. We choose a locally finite countable invariant covering  $G \cdot U_i$  of  $X$  such that  $U_i$  is a slice at  $x(i)$ ,  $i = 1, 2, \dots$ , and there is a  $G_{x(i)}$ -equivalence  $h_i: U_i \times I \times Y_i \rightarrow E|(U_i \times I)$ , where  $Y_i = p^{-1}(x(i))$  (when  $p$  has structure group  $H$ ,  $G_{x(i)}$  acts on the  $H$ -manifold  $Y_i$  by a homomorphism  $G_{x(i)} \rightarrow H$ ).

There is a  $G$ -invariant  $C^\infty$  map  $u_i: X \rightarrow [0, 1]$  such that  $u_i^{-1}(0, 1] \subseteq G \cdot U_i$  and  $\max_i u_i(x) = 1$  for all  $x \in X$ . Define  $G$ -fibre bundle equivalences

$$k_i: G \times_{G_{x(i)}} (U_i \times I \times Y_i) \rightarrow E|(G \cdot U_i \times I)$$

by  $k_i[g, (u, t, y)] = gh_i(u, t, y)$  for  $g \in G$ ,  $u \in U_i$ ,  $t \in I$ ,  $y \in Y_i$ , and define  $G$ -fibre bundle maps

$$\begin{array}{ccc} E & \xrightarrow{F_i} & E \\ \downarrow & & \downarrow \\ X \times I & \xrightarrow{f_i} & X \times I \end{array}$$

as follows:

$$\begin{aligned} f_i(x, t) &= (x, t(1 - u_i(x))), & (x, t) \in X \times I; \\ F_i &= \text{id} \text{ outside } p^{-1}(G \cdot U_i \times I); \\ F_i \circ k_i[g, (u, t, y)] &= k_i[g, (u, t(1 - u_i(u)), y)], \\ [g, (u, t, y)] &\in G \times_{G_{x(i)}} (U_i \times I \times Y_i). \end{aligned}$$

Then  $F = \dots \circ F_3 \circ F_2 \circ F_1$  is a  $G$ -fibre bundle map over  $f = \dots \circ f_3 \circ f_2 \circ f_1$  ( $F, f$  are well-defined since all but a finite number of terms in the infinite compositions are equal to the identity near any point).

By Proposition 2.1,  $E$  is  $G$ -equivalent to  $f^*E$ , which is  $G$ -equivalent to  $(E|X \times 0) \times I$  by the definition of  $f$ . This completes the proof.

Using Theorem 4.1, we can now deduce the equivariant covering homotopy property for a differentiable  $G$ -fibre bundle with structure group any Lie group:

**Theorem 4.2.** *Let  $G$  be a compact Lie group, and  $E \rightarrow X$  a differentiable  $G$ -fibre bundle with structure group a Lie group  $H$ . Then  $E$  is  $G$ -locally trivial.*

*Proof.* Let  $\pi: P \rightarrow X$  be the associated principal bundle, and let  $x \in X$ . The isotropy subgroup  $G_x$  acts on the fibre  $H$  of  $P$  over  $x$  via a homomorphism  $\alpha: G_x \rightarrow H$ . Consider the bundle  $P/\alpha(G_x)$  with fibre  $H/\alpha(G_x)$  associated with  $P$ . The point  $1\alpha(G_x)$  in the fibre over  $x$  is a fixed point for the action of  $G_x$ . Since  $P/\alpha(G_x) \rightarrow X$  is an equivariant submersion onto  $X$ , there is a  $G_x$ -equivariant section  $\sigma$  of  $P/\alpha(G_x)$  defined on some  $G_x$ -invariant neighborhood  $U_x$  of  $x$ , which is  $G_x$ -contractible to  $x$ .

Hence  $E|_{U_x}$  can be reduced to the compact subgroup  $\alpha(G_x)$  of  $H$  so that  $G_x$  still acts as a group of bundle maps. The result now follows from the equivariant covering homotopy property for  $G$ -fibre bundles with compact Lie structure group (Corollary 3.3).

We conclude with some remarks relating  $G$ -local triviality to the equivalence of nearby differentiable actions of a compact Lie group.

If  $p: E \rightarrow X$  is any differentiable  $G$ -fibre bundle with compact fibre, then we can obtain the equivariant covering homotopy property for  $p$  by proving an analogue of Theorem 3.1. Hence  $p$  is also  $G$ -locally trivial.

**Definitions.** Let  $G$  be a compact Lie group, and  $X, Y$  two differentiable manifolds. A differentiable family of actions of  $G$  on  $Y$  parametrized by  $X$  is a differentiable map  $\Phi: X \times G \times Y \rightarrow Y$  such that for each  $x \in X$  the map  $\Phi_x: G \times Y \rightarrow Y$  taking  $(g, y) \in G \times Y$  into  $\Phi(x, g, y)$  is a differentiable action of  $G$  on  $Y$ . This family is said to be *locally trivial at  $x_0 \in X$*  if there are an open neighborhood  $U$  of  $x_0$  in  $X$  and a differentiable map  $\Psi: U \times Y \rightarrow Y$  such that:

1. for each  $x \in U$  the map  $\Psi_x: Y \rightarrow Y$  taking  $y$  into  $\Psi(x, y)$  is a diffeomorphism of  $Y$ , and  $\Psi_{x_0} = \text{id}_Y$ ;

2.  $\Phi(x, g, \Psi(x, y)) = \Psi(x, \Phi(x_0, g, y))$  for each  $x \in U$ ,  $g \in G$ , and  $y \in Y$ .

A family of differentiable actions of  $G$  on  $Y$  is said to be *locally trivial* if it is locally trivial at each point  $x$  of the parameter space  $X$ .

Now if  $p: E \rightarrow X$  is a product bundle  $X \times Y \rightarrow X$  with compact fibre  $Y$ , and  $G$  acts on  $E$  as a group of bundle maps with the induced action on  $X$  trivial, then the  $G$ -local triviality of  $p$  is just a restatement of the fact that a differentiable family of actions of a compact Lie group on a compact manifold  $Y$  is locally trivial (Palais [5], Calabi [1]).

The conjecture of Calabi [1, p. 213] that such a family is locally trivial even when  $Y$  is not compact had already been shown to be false by Palais and Stewart [5], [6]. This shows that a differentiable  $G$ -fibre bundle  $p: E \rightarrow X$  with noncompact fibre does not in general have the equivariant covering homotopy property. (We note here also the observation of Palais and Stewart that a con-

tinuous family of actions of a compact Lie group on a compact space  $Y$  is not in general locally trivial. Hence, though Theorem 4.1, for example, is valid in the continuous case (when  $X$  is paracompact), we cannot hope for an equivariant covering homotopy property for a broad class of continuous  $G$ -fibre bundles.)

From the  $G$ -local triviality of  $G$ -fibre bundles with Lie structure group (Theorem 4.2), we deduce, however, the following result.

**Theorem 4.3.** *Let  $H$  be a Lie group, and  $Y$  an  $H$ -manifold. Let  $\Phi: X \times G \times Y \rightarrow Y$  be a differentiable family of actions of a compact Lie group  $G$  on  $Y$  such that for each  $x \in X$ , there is a homomorphism  $\varphi_x: G \rightarrow H$ , and  $\Phi(x, g, y) = \varphi_x(g)y$  for all  $g \in G, y \in Y$ . Then  $\Phi$  is locally trivial.*

As a final remark we note a type of action of a compact Lie group  $G$  on a differentiable fibre bundle with Lie structure group which is more general than an action as a group of bundle maps and for which the equivariant covering homotopy property still holds. Let  $\pi: P \rightarrow X$  be a differentiable principal bundle with Lie structure group  $H$ , on which  $G$  acts as a group of bundle maps, and let  $Y$  be an effective  $H$ -manifold. If  $G$  acts (on the left) on  $Y$  commuting with the the action of  $H$ , there is an induced action on the total space  $E = P \times_H Y$  of the associated bundle with fibre  $Y$  given by  $g[p, y] = [gp, gy]$ , with respect to which the projection  $E \rightarrow X$  is equivariant. The equivariant covering homotopy property for  $\pi$  clearly implies that for  $E \rightarrow X$ . This also gives a generalization of Theorem 4.3: with the same notation as in Theorem 4.3, if  $G$  acts on  $Y$  commuting with the action of  $H$ , and  $\Phi$  is given by  $\Phi(x, g, y) = \varphi_x(g)y$ , then  $\Phi$  is locally trivial.

### References

- [ 1 ] E. Calabi, *On differentiable actions of compact Lie groups on compact manifolds*, Proc. Conf. on Transformation Groups (New Orleans, 1967), Springer, New York, 1968, 210–213.
- [ 2 ] A. M. Gleason & R. S. Palais, *On a class of transformation groups*, Amer. J. Math. **79** (1957) 631–648.
- [ 3 ] S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.
- [ 4 ] D. Husemoller, *Fibre bundles*, McGraw-Hill, New York, 1968.
- [ 5 ] R. S. Palais, *Equivalence of nearby differentiable actions of a compact group*, Bull. Amer. Math. Soc. **67** (1961) 362–364.
- [ 6 ] R. S. Palais & T. E. Stewart, *Deformations of compact differentiable transformation groups*, Amer. J. Math. **82** (1960) 935–937.
- [ 7 ] G. Segal, *Equivariant K-theory*, Inst. Hautes Études Sci. Publ. Math. No. 34 (1968) 129–151.
- [ 8 ] A. G. Wasserman, *Equivariant differential topology*, Topology **8** (1969) 127–150.

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